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ON A GRAPH PROPERTY GENERALIZING PLANARITY AND FLATNESS

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We introduce a topological graph parameter $\sigma(G)$, defined for any graph G. This parameter characterizes subgraphs of paths, outerplanar graphs, planar graphs, and graphs that have a flat embedding as those graphs G with $\sigma(G) \leq 1, 2, 3$, and 4, respectively. Among several other theorems, we show that if H is a minor of G, then $\sigma(H) \leq \sigma(G)$, that $\sigma(K_n) = n-1$, and that if H is the suspension of G, then $\sigma(H) = \sigma(G)+1$. Furthermore, we show that $\mu(G) \leq \sigma(G)+2$ for each graph G. Here $\mu(G)$ is the graph parameter introduced by Colin de Verdière in [2].

1. Introduction

A graph G is planar if it can be embedded in the plane. An embedding of a graph G in 3-space is flat if for each circuit of G, there exists an open disc in 3-space whose boundary is the circuit, but which is disjoint from the embedding of G; see [12] for more on flat embeddings. What are good analogues of these properties of graphs for higher dimensions? The motivation for this question arises from the invariant $\mu(G)$ introduced by Colin de Verdière [2,3], which characterizes planar graphs as those graphs G with $\mu(G) \leq 3$ [2], and graphs that have a flat embedding as those graphs G with $\mu(G) \leq 4$ [8]. In this paper, we introduce higher dimensional analogues of planarity and flatness and give upper bounds on $\mu(G)$ for graphs satisfying these mapping properties. The definition is by means of cell complexes whose 1-skeleton is G, and their intersection properties. In [5] algebraic characteri-

Mathematics Subject Classification (2000): 05C10, 05C50, 05C83 zations were given for planar graphs and graphs that have a flat embedding. Work in this paper extends these results to higher dimensions.

We call a continuous mapping ϕ from a cell complex $\mathcal{C}$ into n-space *proper* if $\phi(\sigma_1)\cap\phi(\sigma_2)=\emptyset$ for every pair of nonadjacent cells σ_1,σ_2 with $\dim\sigma_1+\dim\sigma_2\leq n-1$. A mapping ϕ from a cell complex $\mathcal{C}$ into n-space is called *even* if it is proper and the intersection number of $\phi(\sigma_1)$ and $\phi(\sigma_2)$ is even for every pair of nonadjacent cells σ_1,σ_2 of $\mathcal{C}$ with $\dim\sigma_1+\dim\sigma_2=n$. (See Section 3 for the definitions of adjacency of cells and intersection number.) We define $\sigma(G)$ as the smallest integer $n\geq 0$ such that every cell complex whose 1-skeleton is G has an even mapping in $\mathbb{R}^n$. This graph invariant has the property that $\sigma(H)\leq\sigma(G)$ if H is a minor of G. We will show that

- $\sigma(G)=0$ if and only if G contains exactly one vertex,
- $\sigma(G)\leq 1$ if and only if G is a subgraph of a path,
- $\sigma(G)\leq 2$ if and only if G is outerplanar,
- $\sigma(G)\leq 3$ if and only if G is planar, and
- $\sigma(G)\leq 4$ if and only if G has a flat embedding in $\mathbb{R}^3$.

Furthermore, if H is obtained from G by adding a new vertex adjacent to all vertices in G, then $\sigma(H)=\sigma(G)+1$ unless G is the complement of K_2. Notice that for $k\leq 4$, $\mu(G)\leq k$ if and only if $\sigma(G)\leq k$.

We show that $\sigma(G)\leq\mu(G)+2$ for any graph G. The proof of this uses the graph parameter $\lambda(G)$, introduced by van der Holst, Laurent, and Schrijver [6]. We show that $\lambda(G)\leq\sigma(G)$. The proof of this follows in part the proof presented by Lovász and Schrijver [8]. Then, using a result of Pendavingh [10] which says that $\mu(G)\leq\lambda(G)+2$, we obtain that $\mu(G)\leq\sigma(G)+2$ for any graph G.

The outline of the paper is as follows. In Section 2, we recall some notions from algebraic topology. In Section 3, we give the definition of even mappings of cell complexes and of the delete product of a cell complex. We give a criterion for a cell complex to have an even mapping in $\mathbb{R}^n$, $n>1$. In Section 4, we study antipodal cell complexes and equivariant maps. We introduce the notion of nonadjacency maps and of symmetric n-cycles of the deleted product of a cell complex. This leads to a criterion for a cell complex to have an even mapping in $\mathbb{R}^n$ (with $n\geq 0$ an integer). This criterion says that a cell complex has an even mapping in $\mathbb{R}^n$ if and only if each symmetric n-cycle of the deleted product of this cell complex has a certain covering property. In Section 5, we introduce a special type of cell complexes associated with graphs, called closures of graphs, and show that each cell complex whose 1-skeleton is G has a nonadjacency preserving map into a closure of G. In Section 6, we introduce the new graph parameter $\sigma(G)$. We show that this parameter is minor-monotone and that it does not increase when we apply a ΔY-transformation on G. Furthermore, we show that, with one exception, $\sigma(H)=\sigma(G)+1$ if H is the graph obtained from G by adding a new vertex and connecting this to each vertex of G by an edge. In Section 7, we prove the above given characterizations of $\sigma(G)\leq k$ for $k\in\{0,1,\ldots,4\}$. In Section 8, we show that $\mu(G)\leq\sigma(G)+2$ for each graph G.

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2. Preliminaries

We denote the n-ball by B^n and the n-sphere by S^n.

A topological space X is *n-connected* if, for $i=0,1,\ldots,n$, each continuous map $S^i\to X$ extends to a continuous map $B^{i+1}\to X$. This is equivalent to $\pi_i(X)$ is trivial for $i=0,\ldots,n$.

If (X,A) and (Y,B) are pairs, a *map of pairs* $f\colon (X,A)\to(Y,B)$ is a map $f\colon X\to Y$ for which $f(A)\subseteq B$.

A *cell complex* $\mathcal{C}$ is a topological space which can recursively be constructed as follows. Start with a finite set $\mathcal{C}^0$. The points of $\mathcal{C}^0$ are regarded as 0-cells. Suppose now that $\mathcal{C}^{n-1}$ has been constructed. Then construct $\mathcal{C}^n$ from $\mathcal{C}^{n-1}$ by attaching a finite number of n-dimensional balls B^n_α via continuous maps $f_\alpha\colon \partial B^n_\alpha\to\mathcal{C}^{n-1}$. This means that $\mathcal{C}^n$ is the quotient space of the disjoint union of $\mathcal{C}^{n-1}$ with a collection of n-balls B^n_α under the identifications $x\sim f_\alpha(x)$ for $x\in\partial B^n_\alpha$. Set $\mathcal{C}=\cup_{k=0}^{\infty}\mathcal{C}^k$. An *$n$-cell* of $\mathcal{C}$ is a component of $\mathcal{C}^n\setminus\mathcal{C}^{n-1}$. So each n-cell is the interior of an n-ball B^n_α. If σ is an n-cell of $\mathcal{C}^n$ which is the interior of B^n_α, we also write f_α as f_σ. The maps f_α are called *attaching maps* and $\mathcal{C}^n$ is called the *n-skeleton* of the cell complex $\mathcal{C}$. A *subcomplex* of a cell complex $\mathcal{C}$ is a closed subspace of $\mathcal{C}$ that is a union of cells of $\mathcal{C}$. If $\mathcal{C}$ has finite dimension, we call $\mathcal{C}$ a finite cell complex.

Let $\mathcal{C}$ and $\mathcal{D}$ be cell complexes. A continuous map $g\colon\mathcal{C}\to\mathcal{D}$ is a *cellular map* if $g(\mathcal{C}^n)\subseteq\mathcal{D}^n$ for each $n\geq 0$. Continuous maps can be approximated by cellular maps.

Theorem 1 (Cellular Approximation Theorem). *Let $\mathcal{C}$ and $\mathcal{D}$ be cell complexes and let $g\colon\mathcal{C}\to\mathcal{D}$ be a continuous map. Then g is homotopic to a cellular map.*

See Bredon [1] for a proof of this theorem.

For a cell complex $\mathcal{C}$, the *group of n-chains* of $\mathcal{C}$ (with coefficients in $\mathbb{Z}_2$), denoted by $\overline{C}_n(\mathcal{C})$, is the nth singular homology group $H_n(\mathcal{C}^n,\mathcal{C}^{n-1};\mathbb{Z}_2)$ of the pair $(\mathcal{C}^n,\mathcal{C}^{n-1})$. See Massey [9] and Bredon [1] for the notation and terminology used in algebraic topology. The attaching map f_α induces a map of pairs

$$F_\alpha\colon (B^n_\alpha,\partial B^n_\alpha)\to(\mathcal{C}^n,\mathcal{C}^{n-1}),$$

and so it induces a homomorphism

$$(F_\alpha)_n\colon H_n(B^n_\alpha,\partial B^n_\alpha;\mathbb{Z}_2)\to H_n(\mathcal{C}^n,\mathcal{C}^{n-1};\mathbb{Z}_2).$$

The homology group $H_n(B^n_\alpha,\partial B^n_\alpha;\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$; let γ be the nonzero homology class in $H_n(B^n_\alpha,\partial B^n_\alpha;\mathbb{Z}_2)$. If σ is an n-cell of $\mathcal{C}^n$ corresponding to α, then we denote the n-chain $(F_\alpha)_n(\gamma)$ by σ. Each n-chain c is therefore of the form $\sum_{\sigma\in S}\sigma$, where S is a set of n-cells of $\mathcal{C}^n$. So we may view an n-chain of $\mathcal{C}$ as a set of n-cells of $\mathcal{C}$, and we will use both these point of views. The addition operator in $\overline{C}_n(\mathcal{C})$ then corresponds to the symmetric difference of two n-chains.

The boundary operator

$$d_n\colon \overline{C}_n(\mathcal{C})\to\overline{C}_{n-1}(\mathcal{C})$$

is defined as $j_{n-1}\circ\partial_n$, where

$$\partial_n\colon H_n(\mathcal{C}^n,\mathcal{C}^{n-1};\mathbb{Z}_2)\to H_{n-1}(\mathcal{C}^{n-1};\mathbb{Z}_2)$$

is the boundary operator of the pair $(\mathcal{C}^n,\mathcal{C}^{n-1})$ and

$$j_{n-1}\colon H_{n-1}(\mathcal{C}^{n-1};\mathbb{Z}_2)\to H_{n-1}(\mathcal{C}^{n-1},\mathcal{C}^{n-2};\mathbb{Z}_2)$$

is the homomorphism induced by the inclusion map; see [9, page 84]. The boundary operator satisfies $d_n\circ d_{n+1}=0$. The group of *n-cycles*, $\overline{Z}_n(\mathcal{C})$, of $\mathcal{C}$ is defined as the kernel of d_n. The homomorphism j_{n-1} is one-to-one and the image of j_{n-1} is equal to $\overline{Z}_{n-1}(\mathcal{C})$.

If $\mathcal{C}$ is a graph $G=(V,E)$ (viewed as a topological space), then $\overline{C}_1(G)$ is the group $\mathbb{Z}_2^E$ and $\overline{Z}_1(G)$ is the cycle space of G.

If $\mathcal{C}$ and $\mathcal{D}$ are cell complexes, then $\mathcal{C}\times\mathcal{D}$ is a cell complex. For a k-chain c_1 of $\mathcal{C}$ and an l-chain c_2 of $\mathcal{D}$, $c_1\times c_2$ is the $(k+l)$-chain defined by

$$c_1\times c_2=\sum_{\sigma\in c_1}\sum_{\tau\in c_2}\sigma\times\tau.$$

If c_1 is a k-chain of $\mathcal{C}$ and c_2 is an l-chain of $\mathcal{D}$, then

$$d_{k+l}(c_1\times c_2)=d_k(c_1)\times c_2+c_1\times d_l(c_2).$$

An n-cochain c of $\mathcal{C}$ is a linear mapping $c\colon \overline{C}_n(\mathcal{C}) \to \mathbb{Z}_2$. The vector space of all n-cochains is denoted by $\overline{C}^n(\mathcal{C})$. The coboundary operator $\delta_{n-1}\colon \overline{C}^{n-1}(\mathcal{C}) \to \overline{C}^n(\mathcal{C})$ is defined by $\delta_{n-1}(a)(b) = a(d_n(b))$ for all $a \in \overline{C}^{n-1}(\mathcal{C})$ and all n-chains b of $\mathcal{C}$.

If $g\colon \mathcal{C} \to \mathcal{D}$ is a cellular map, then, for each integer $n \geq 0$, there are homomorphisms

$$g_n\colon \overline{C}_n(\mathcal{C}) \to \overline{C}_n(\mathcal{D})$$

induced by g. These homomorphisms are *chain maps*, which means that

$$d_n \circ g_n = g_{n-1} \circ d_n \tag{1}$$

for each integer $n > 0$. In particular,

$$g_n(\overline{Z}_n(\mathcal{C})) \subseteq \overline{Z}_n(\mathcal{D})$$

for each integer $n \geq 0$.

Let $\mathcal{C}$ be a cell complex and let $f\colon (\mathcal{C}^n, \mathcal{C}^{n-1}) \to (\mathbb{R}^n, \mathbb{R}^n - \{0\})$ be a continuous map of pairs. Let σ be an n-cell of $\mathcal{C}$ and let $S = \{p \in \sigma \mid f(p) = 0\}$. We say that f is in generic position at σ if S is a finite set and there is a collection of neighborhoods $\{U_p \subseteq \sigma : p \in S\}$ such that $U_p \cap U_q = \emptyset$ for all $p, q \in S$ with $p \neq q$ and $f(U_p)$ is a neighborhood of $0 \in \mathbb{R}^n$ for all $p \in S$. If f is in generic position at σ, the *covering number* of σ under f is defined to be 0 if $|S|$ is even and 1 if $|S|$ is odd. Also if f is not in generic position we can define the covering number. This goes as follows. The map f induces a homomorphism $f_n\colon \overline{C}_n(\mathcal{C}) = H_n(\mathcal{C}^n, \mathcal{C}^{n-1}; \mathbb{Z}_2) \to H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}_2)$. Let β be the nonzero homology class of $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}_2)$. For any n-chain c of $\mathcal{C}$, we define the *covering number* of c under f, $\mathrm{Cov}_2(f, c) \in \mathbb{Z}_2$, by

$$f_n(c) = \mathrm{Cov}_2(f, c)\beta.$$

If $g\colon \mathcal{D} \to \mathcal{C}$ is a cellular map and c is an n-chain of $\mathcal{D}$, then $\mathrm{Cov}_2(f \circ g, c) = \mathrm{Cov}_2(f, g_n(c))$.

Give the n-sphere S^n the following cell structure $\mathcal{S}^n$ with exactly two cells in each dimension k, $0 \leq k \leq n$, by letting the k-cells be the two hemispheres of $S^k \subset S^n$ for each k. We denote by s_k the k-cycle of $\mathcal{S}^n$ consisting of the two k-cells.

For any cellular map $f\colon \mathcal{C} \to \mathcal{S}^n$ and any n-cycle z of $\mathcal{C}$, we define $\mathrm{Deg}_2(f, z) \in \mathbb{Z}_2$ by

$$f_n(z) = \mathrm{Deg}_2(f, z) s_n.$$

Define $J\colon \mathbb{R}^n - \{0\} \to S^{n-1}$ by $J(x) = x/\|x\|$.

Lemma 2. *Let $\mathcal{C}$ be a cell complex and let $f\colon (\mathcal{C}^n, \mathcal{C}^{n-1}) \to (\mathbb{R}^n, \mathbb{R}^n - \{0\})$ be a continuous map of pairs and let f' be the restriction of f to $\mathcal{C}^{n-1}$. If $\phi\colon \mathcal{C}^{n-1} \to \mathcal{S}^{n-1}$ is a cellular map homotopic to $J \circ f'$, then* $\mathrm{Deg}_2(\phi, d_n(c)) = \mathrm{Cov}_2(f, c)$ *for each n-chain c of $\mathcal{C}$.*

Proof. Let $f_n\colon \overline{C}_n(\mathcal{C}) \to H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}_2)$ be the homomorphism induced by f, and let $f'_{n-1}\colon H_{n-1}(\mathcal{C}^{n-1}; \mathbb{Z}_2) \to H_{n-1}(\mathbb{R}^n - \{0\}; \mathbb{Z}_2)$ be the homomorphism induced by f'. Let $J_{n-1}\colon H_{n-1}(\mathbb{R}^n - \{0\}; \mathbb{Z}_2) \to H_{n-1}(S^{n-1}; \mathbb{Z}_2)$ be the homomorphism induced by J.

Let β be the generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}_2)$. Then

$$f_n(c) = \mathrm{Cov}_2(f, c)\beta. \tag{2}$$

Applying $j_{n-1} \circ J_{n-1} \circ \partial_n$ to both sides of (2) yields

$$j_{n-1}(J_{n-1}(\partial_n(f_n(c)))) = \mathrm{Cov}_2(f, c) j_{n-1}(J_{n-1}(\partial_n(\beta))). \tag{3}$$

Using that $j_{n-1}(J_{n-1}(\partial_n(\beta))) = s_{n-1}$, and using that $\partial_n(f_n(c)) = f'_{n-1}(\partial_n(c))$, we obtain

$$j_{n-1}(J_{n-1}(f'_{n-1}(\partial_n(c)))) = \mathrm{Cov}_2(f, c) s_{n-1}.$$

Let $\Phi\colon \mathcal{C}^n \to \mathcal{S}^n$ be a cellular map (which we view as a continuous map of pairs $\Phi\colon (\mathcal{C}^n, \mathcal{C}^{n-1}) \to (\mathcal{S}^n, \mathcal{S}^{n-1})$) such that its restriction to $\mathcal{C}^{n-1}$, which we denote by Φ', is homotopic to $J \circ f'$. Then $J_{n-1}(f'_{n-1}(\partial_n(c))) = \Phi'_{n-1}(\partial_n(c))$. Since $\Phi'_{n-1}(\partial_n(c)) = \partial_n(\Phi_n(c))$, we obtain $j_{n-1}(\partial_n(\Phi_n(c))) = \mathrm{Cov}_2(f, c) s_{n-1}$. Using that $j_{n-1} \circ \partial_n = d_n$, we obtain

$$\Phi_{n-1}(d_n(c)) = d_n(\Phi_n(c)) = \mathrm{Cov}_2(f, c) s_{n-1}.$$

Hence $\mathrm{Cov}_2(f, c) = \mathrm{Deg}_2(\Phi, d_n(c))$. ∎

3. Even mappings

Let $\mathcal{C}$ be a cell complex. We call two cells σ_1, σ_2 of $\mathcal{C}$ *adjacent* if the smallest subcomplexes of $\mathcal{C}$ containing σ_1 and σ_2, respectively, have nonempty intersection. Recall that a continuous mapping ϕ from $\mathcal{C}$ into n-space is *proper* if $\phi(\sigma_1) \cap \phi(\sigma_2) = \emptyset$ for every pair of nonadjacent cells σ_1, σ_2 with $\dim \sigma_1 + \dim \sigma_2 \leq n - 1$. Let $\phi\colon \mathcal{C} \to \mathbb{R}^n$ be a proper map and let σ_1, σ_2 be a pair of nonadjacent cells with $\dim \sigma_1 + \dim \sigma_2 = n$. If $\phi(\sigma_1)$ and $\phi(\sigma_2)$ are in generic position, that is, if $\phi(\sigma_1)$ and $\phi(\sigma_2)$ have a finite number of intersection points and at these points they intersect transversally, then the mod 2 intersection number of σ_1 and σ_2 under ϕ is the reduction modulo 2 of the number of intersections points. The notion of intersection number can be extended to the case where $\phi(\sigma_1)$ and $\phi(\sigma_2)$ need not be in generic position. To this end, we first recall the *deleted product* of a cell complex $\mathcal{C}$. This is the subcomplex of $\mathcal{C}\times\mathcal{C}$ consisting of all cells $\sigma_1\times\sigma_2$ with σ_1 and σ_2 nonadjacent. Denote the deleted product of $\mathcal{C}$ by $D(\mathcal{C})$ and define

$$R(\phi)\colon D(\mathcal{C}) \to \mathbb{R}^n$$

by $R(\phi)(p_1,p_2):=\phi(p_1)-\phi(p_2)$. Since ϕ is proper, $0\notin R(\phi)(D(\mathcal{C})^{n-1})$. Now, if $\phi(\sigma_1)$ and $\phi(\sigma_2)$ have a finite number of intersection points and at these points they intersect transversally, then the reduction modulo 2 of the number of intersection points is equal to the reduction modulo two of the number of points $(x,y)\in\sigma_1\times\sigma_2$ such that $R(\phi)(x,y)=0$, which is equal to $\mathrm{Cov}_2(R(\phi),\sigma_1\times\sigma_2)$. For the general case we define the *mod 2 intersection number*, $I_2(\phi;\sigma_1,\sigma_2)$, of σ_1 and σ_2 under ϕ by

$$I_2(\phi;\sigma_1,\sigma_2)=\mathrm{Cov}_2(R(\phi),\sigma_1\times\sigma_2).$$

If $I_2(\phi;\sigma_1,\sigma_2)=0$ for each pair of nonadjacent cells with $\dim\sigma_1+\dim\sigma_2=n$, then we say that ϕ is an *even map*.

Denote by $\widetilde{D(\mathcal{C})}$ the cell complex obtained from $D(\mathcal{C})$ by identifying (x,y) with (y,x) for all $(x,y)\in D(\mathcal{C})$, and let $p\colon D(\mathcal{C})\to\widetilde{D(\mathcal{C})}$ denote the projection. Clearly, p is a cellular map. For each cell $\sigma_1\times\sigma_2$ of $D(\mathcal{C})$, $p(\sigma_1\times\sigma_2)$ is a cell of $\widetilde{D(\mathcal{C})}$. For any proper mapping $\phi\colon\mathcal{C}\to\mathbb{R}^n$, we define $J_\phi\in\overline{C}^n(\widetilde{D(\mathcal{C})})$ by $J_\phi(p_n(\sigma_1\times\sigma_2))=\mathrm{Cov}_2(R(\phi),\sigma_1\times\sigma_2)$ for each n-cell $\sigma_1\times\sigma_2$ of $D(\mathcal{C})$. (J_ϕ is well-defined as $\mathrm{Cov}_2(R(\phi),\sigma_1\times\sigma_2)=\mathrm{Cov}_2(R(\phi),\sigma_2\times\sigma_1)$.) Then J_ϕ equals zero if and only if ϕ is an even map. In the next section we prove the following proposition.

Proposition 3. *Let $\mathcal{C}$ be a cell complex. For each n-cycle $\tilde{z}$ of $\widetilde{D(\mathcal{C})}$, $J_\phi(\tilde{z})$ is independent of the proper map $\phi\colon\mathcal{C}\to\mathbb{R}^n$.*

Lemma 4. *Let $\mathcal{C}$ be a cell complex and let $n\geq 0$ be an integer. If $\phi\colon\mathcal{C}\to\mathbb{R}^n$ is an even map, then $J_\phi(\tilde{c})=0$ for each n-chain $\tilde{c}$ of $\widetilde{D(\mathcal{C})}$.*

Proof. Since ϕ is an even map, $\mathrm{Cov}_2(R(\phi),\sigma_1\times\sigma_2)=0$ for each n-cell $\sigma_1\times\sigma_2$ of $D(\mathcal{C})$. Let c be an n-chain of $D(\mathcal{C})$ such that $\tilde{c}=p_n(c)$. Then $J_\phi(\tilde{c})=\mathrm{Cov}_2(R(\phi),c)=0$. ∎

In particular, if $\phi\colon\mathcal{C}\to\mathbb{R}^n$ is an even map, then $J_\phi(\tilde{z})=0$ for each n-cycle $\tilde{z}$ of $\widetilde{D(\mathcal{C})}$. The converse of this is true for cell complexes whose underlying topological space is a simplicial complex. For this we use a theorem of Wu [15, Theorem 7]. (Although a simplicial complex is itself a cell complex, the cells of cell complexes we consider may consist of many simplices.)

Theorem 5 **([15]).** *Let $\mathcal{C}$ be a cell complex whose underlying topological space is a simplicial complex, and let $n > 1$ be an integer. Let $\phi\colon \mathcal{C} \to \mathbb{R}^n$ be a simplicial map in generic position. Then, for each $(n-1)$-cochain c of $\widetilde{D(\mathcal{C})}$, there exists a simplicial map $\phi'\colon \mathcal{C} \to \mathbb{R}^n$ in generic position such that $J_{\phi'} - J_\phi = \delta_{n-1}(c)$.*

Theorem 6. *Let $\mathcal{C}$ be a cell complex whose underlying topological space is a simplicial complex, and let $n > 1$ be an integer. Let $\psi\colon \mathcal{C} \to \mathbb{R}^n$ be a simplicial map in generic position. If $J_\psi(\tilde{z}) = 0$ for each n-cycle $\tilde{z}$ of $\widetilde{D(\mathcal{C})}$, then there is an even map $\phi\colon \mathcal{C} \to \mathbb{R}^n$.*

Proof. Since $J_\psi(\tilde{z}) = 0$ for each n-cycle $\tilde{z}$ of $\widetilde{D(\mathcal{C})}$, there is an $(n-1)$-cochain c such that $J_\psi = \delta_{n-1}(c)$. By Theorem 5 there exists a simplicial mapping $\phi\colon \mathcal{C} \to \mathbb{R}^n$ in generic position such that $J_\phi - J_\psi = \delta_{n-1}(c)$. Hence $J_\phi = 0$, so ϕ is an even map. ∎

4. Equivariant mappings

An *antipodal cell complex* is a pair $(\mathcal{C}, T)$ with $\mathcal{C}$ a cell complex and $T\colon \mathcal{C} \to \mathcal{C}$ a cellular map with the properties

(i) $T(T(x)) = x$ for all $x \in \mathcal{C}$, and
(ii) for each cell σ, $T(\sigma) \cap \sigma = \emptyset$.

The map T induces a chain map $T_n\colon \overline{C}_n(\mathcal{C}) \to \overline{C}_n(\mathcal{C})$ for each integer $n \geq 0$. An n-cycle z of $\mathcal{C}$ is *symmetric* if $T_n(z) = z$. It is easy to check that any symmetric n-cycle z is of the form $c + T_n(c)$ where c is an n-chain of $\mathcal{C}$.

Lemma 7. *If c is an $(n+1)$-chain such that $c + T_{n+1}(c)$ is a symmetric $(n+1)$-cycle, then $d_{n+1}(c)$ is a symmetric n-cycle.*

Proof. Let $z = c + T_{n+1}(c)$. Then $d_{n+1}(c) + T_n(d_{n+1}(c)) = d_{n+1}(c + T_{n+1}(c)) = d_{n+1}(z) = 0$, so $d_{n+1}(c)$ is a symmetric n-cycle. ∎

Homological properties of antipodal cell complexes were studied by Richardson [11] and Smith [14].

Examples of antipodal cell complexes are the $(n-1)$-skeletons of centrally symmetric polytopes M in $\mathbb{R}^n$, with $T\colon M^{n-1} \to M^{n-1}$ defined by $T(x) = -x$. Also $\mathcal{S}^n$, the cell complex homeomorphic to S^n, which we introduced in Section 2, can be made an antipodal cell complex $(\mathcal{S}^n, T)$ by defining $T(x) = -x$. The unique nonzero n-cycle s_n of $\mathcal{S}^n$ is symmetric. Another example is the deleted product $D(\mathcal{C})$ of a cell complex $\mathcal{C}$. Define $T\colon D(\mathcal{C}) \to D(\mathcal{C})$ by

$T(p_1,p_2)=(p_2,p_1)$ for all $(p_1,p_2)\in D(\mathcal{C})$. Then $(D(\mathcal{C}),T)$ is an antipodal cell complex.

If $(\mathcal{C},T)$ is an antipodal cell complex, we denote by $\widetilde{\mathcal{C}}$ the cell complex obtained by identifying each $x\in\mathcal{C}$ with $T(x)$. Let $p\colon \mathcal{C}\to\widetilde{\mathcal{C}}$ be the projection. This means that for each $x\in\mathcal{C}$, $p(x)$ is the equivalence class of x in $\widetilde{\mathcal{C}}$ (and hence $p(x)=p(T(x)))$. p is a cellular map, and hence p_n is defined. Moreover, if z is a symmetric n-cycle of $(\mathcal{C},T)$ and c is an n-chain of $\mathcal{C}$ such that $c+T_n(c)=z$, then $p_n(c)$ is an n-cycle of $\widetilde{\mathcal{C}}$. Conversely, if $\tilde{z}$ is an n-cycle of $\widetilde{\mathcal{C}}$, then there is a unique symmetric n-cycle z of $(\mathcal{C},T)$ for which there exists an n-chain c of $\mathcal{C}$ with $z=c+T_n(c)$ such that $p_n(c)=\tilde{z}$.

Let $(\mathcal{C},T)$ and $(\mathcal{D},R)$ be antipodal cell complexes. An *equivariant* map $f\colon (\mathcal{C},T)\to(\mathcal{D},R)$ is a continuous map satisfying $f\circ T=R\circ f$.

Lemma 8. *Let $(\mathcal{C},T)$ and $(\mathcal{D},R)$ be antipodal cell complexes, where $\mathcal{D}$ is arcwise connected. Then each equivariant map $f\colon (\mathcal{C},T)\to(\mathcal{D},R)$ is homotopic to an equivariant cellular map $g\colon (\mathcal{C},T)\to(\mathcal{D},R)$.*

Proof. Let $p\colon \mathcal{C}\to\widetilde{\mathcal{C}}$ and $q\colon \mathcal{D}\to\widetilde{\mathcal{D}}$ be the projections induced by T and R, respectively. Let $\tilde{f}\colon \widetilde{\mathcal{C}}\to\widetilde{\mathcal{D}}$ be the map induced by f ($\tilde{f}$ is continuous and its existence follows from the equivariance of f). By the Cellular Approximation Theorem there is a homotopy $\tilde{F}\colon \widetilde{\mathcal{C}}\times I\to\widetilde{\mathcal{D}}$ such that $\tilde{F}(x,0)=\tilde{f}(x)$ for all $x\in\widetilde{\mathcal{C}}$ and the continuous mapping $\tilde{g}\colon \widetilde{\mathcal{C}}\to\widetilde{\mathcal{D}}$ defined by $\tilde{g}(x)=\tilde{F}(x,1)$ for all $x\in\mathcal{C}$ is cellular. Define $F\colon \mathcal{C}\times I\to\widetilde{\mathcal{D}}$ by $F(x,t)=\tilde{F}(p(x),t)$. Then $q(f(x))=F(x,1)$ for all $x\in\mathcal{C}$. By the Covering Homotopy Theorem (see [1, pages 140, 141]) there is a homotopy $G\colon \mathcal{C}\times I\to\mathcal{D}$ such that $G(x,0)=f(x)$ for all $x\in\mathcal{C}$, and $q\circ G=F$. Define $g\colon \mathcal{C}\to\mathcal{D}$ by $g(x)=G(x,1)$, $x\in\mathcal{C}$. Then g is an equivariant cellular map homotopic to f. ∎

If $f\colon (\mathcal{C},T)\to(\mathcal{D},R)$ is an equivariant cellular map, then $f_n\colon \overline{C}_n(\mathcal{C})\to\overline{C}_n(\mathcal{D})$ satisfies $f_n\circ T_n=R_n\circ f_n$.

Lemma 9. *Let $f\colon (\mathcal{C},T)\to(\mathcal{D},R)$ be an equivariant cellular map and $f_n\colon \overline{C}_n(\mathcal{C})\to\overline{C}_n(\mathcal{D})$ be the chain map induced by f. Then $f_n(z)$ is a symmetric n-cycle of $(\mathcal{D},R)$ for any symmetric n-cycle z of $(\mathcal{C},T)$.*

Proof. Since f is equivariant, $f_n\circ T_n=R_n\circ f_n$. Let c be an n-chain of $\mathcal{C}$ such that $z=c+T_n(c)$. Then $f_n(z)$ is a symmetric n-chain as $f_n(c+T_n(c))=f_n(c)+f_n(T_n(c))=f_n(c)+R_n(f_n(c))$, and it is an n-cycle as $d_n(f_n(z))=f_{n-1}(d_n(z))=0$. ∎

Lemma 10. *Let $n>0$ be an integer. Let $f\colon (\mathcal{C},T)\to(\mathcal{S}^n,R)$ be an equivariant cellular map and let z be a symmetric n-cycle of $(\mathcal{C},T)$. If c is an n-chain of $\mathcal{C}$ such that $z=c+T_n(c)$, then* $\mathrm{Deg}_2(f,d_n(c))=\mathrm{Deg}_2(f,z)$.

Proof. Let

$$f_n(c) = a_1\sigma_1 + a_2\sigma_2, \tag{4}$$

where σ_1, σ_2 are the n-cells of $\mathcal{S}^n$. Then $f_n(z) = f_n(c+T_n(c)) = (a_1+a_2)(\sigma_1+\sigma_2) = (a_1+a_2)s_n$, and so $\mathrm{Deg}_2(f,z) = a_1+a_2$. Applying the boundary operator to both sides of (4), we get $f_{n-1}(d_n(c)) = d_n(f_n(c)) = d_n(a_1\sigma_1 + a_2\sigma_2) = (a_1+a_2)s_{n-1} = \mathrm{Deg}_2(f,z)s_{n-1}$. ∎

Lemma 11. *Let $n \geq 0$ be an integer. Let $f, g\colon (\mathcal{C}, T) \to (\mathcal{S}^n, R)$ be equivariant cellular maps. Let z be a symmetric n-cycle of $(\mathcal{C}, T)$. Then $\mathrm{Deg}_2(f,z) = \mathrm{Deg}_2(g,z)$.*

Proof. If c is a 0-chain, then both $\mathrm{Deg}_2(f, c+T_0(c))$ and $\mathrm{Deg}_2(g, c+T_0(c))$ equal the size of the support of c mod 2. Induction on n shows that $\mathrm{Deg}_2(f,z) = \mathrm{Deg}_2(g,z)$ for any symmetric n-cycle z of $(\mathcal{C}, T)$. ∎

Let $(\mathcal{C}, T)$ be an antipodal cell complex. A continuous map of pairs $f\colon (\mathcal{C}^n, \mathcal{C}^{n-1}) \to (\mathbb{R}^n, \mathbb{R}^n - \{0\})$ is *equivariant* if $f(T(x)) = -f(x)$ for all $x \in \mathcal{C}^n$. If z is a symmetric n-cycle of $\mathcal{C}$ and $f\colon (\mathcal{C}^n, \mathcal{C}^{n-1}) \to (\mathbb{R}^n, \mathbb{R}^n - \{0\})$ is an equivariant map of pairs, we define

$$I(f,z) = \mathrm{Cov}_2(f,c),$$

where c is any n-chain such that $z = c + T_n(c)$. It is easy to see that $I(f,z)$ is well-defined; that is, it is independent of the choice of c.

If $n = 0$, $\mathrm{Cov}_2(f,c)$ equals the size of the support of c mod 2. So, if $n = 0$, then $I(f,z)$ is independent of the equivariant map. Also if $n > 0$, $I(f,z)$ is independent of the equivariant map f. To see this, let f' be the restriction of f to $\mathcal{C}^{n-1}$ and define $\phi = J \circ f'$, where $J\colon \mathbb{R}^n - \{0\} \to \mathcal{S}^{n-1}$ is defined by $J(x) = x/\|x\|$. Then ϕ is an equivariant map. Let $\Phi\colon \mathcal{C}^{n-1} \to \mathcal{S}^{n-1}$ be an equivariant cellular map homotopic to ϕ (by Lemma 8 such an equivariant cellular map exists). If c is an n-chain of $\mathcal{C}$ such that $z = c + T_n(c)$, then $I(f,z) = \mathrm{Deg}_2(\Phi, d_n(c))$, by Lemma 2. By Lemma 11, $\mathrm{Deg}_2(\Phi, d_n(c))$ is independent of the equivariant map Φ, so $I(f,z)$ is independent of the equivariant map f. Therefore, we can define

$$I(z) = I(f,z)$$

for any symmetric n-cycle z of an antipodal cell complex $(\mathcal{C}, T)$. (The existence of an equivariant map $f\colon \mathcal{C} \to \mathbb{R}^n$ can be shown by induction over the skeletons of $\mathcal{C}$.)

From Lemma 10, we obtain:

Lemma 12. *Let* $(\mathcal{C},T)$ *be an antipodal cell complex and let* $n>0$ *be an integer. Let* z *be a symmetric* n*-cycle of* $\mathcal{C}$*. If* c *is an* n*-chain of* $\mathcal{C}$ *such that* $z=c+T_n(c)$*, then* $I(z)=I(d_n(c))$*.*

Theorem 13 (Borsuk). $I(s_n)=1$.

Proof. Apply induction on n. ∎

Let $\widetilde{\mathcal{C}}$ be the cell complex obtained from $\mathcal{C}$ by identifying x and $T(x)$. For any n-cycle $\tilde{z}$ of $\widetilde{\mathcal{C}}$, there is a unique symmetric n-cycle z of $(\mathcal{C},T)$ for which there exists an n-chain c of $\mathcal{C}$ with $z=c+T_n(c)$ and $p_n(c)=\tilde{z}$. We define

$$J(\tilde{z}) = I(z).$$

In the previous section we defined $J_\phi \in \overline{C}^n(\widetilde{D(\mathcal{C})})$. From the independence of $I(f,z)$ of the equivariant map f, it follows that for any n-cycle $\tilde{z}$ of $\widetilde{D(\mathcal{C})}$, $J_\phi(\tilde{z}) = J(\tilde{z})$. Furthermore, Lemma 12 provides a way to compute $J(\tilde{z})(= J_\phi(\tilde{z}))$ for an n-cycle $\tilde{z}$ of $\widetilde{D(\mathcal{C})}$.

Lemma 14. *Let* $g\colon (\mathcal{C},T)\to(\mathcal{D},R)$ *be an equivariant cellular map. Then* $I(z)=I(g_n(z))$ *for each symmetric* n*-cycle* z *of* $\mathcal{C}$*.*

Proof. Let $f\colon (\mathcal{C}^n,\mathcal{C}^{n-1})\to(\mathbb{R}^n,\mathbb{R}^n-\{0\})$ be an equivariant map of pairs. Then $I(z)=I(f\circ g,z)=I(f,g_n(z))=I(g_n(z))$ for each symmetric n-cycle z of $\mathcal{C}$. ∎

Let $\mathcal{C}$ and $\mathcal{D}$ be cell complexes. We say that a cellular map $f\colon \mathcal{C}\to\mathcal{D}$ is *nonadjacency preserving* if for each pair of nonadjacent cells σ_1,σ_2 of $\mathcal{C}$, $f(\sigma_1)$ and $f(\sigma_2)$ are nonadjacent. For any such map we define the map

$$D(f)\colon D(\mathcal{C}) \to D(\mathcal{D})$$

by $D(f)(p,q)=(f(p),f(q))$. Then $D(f)$ is an equivariant cellular map. From Lemma 14, we obtain:

Lemma 15. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be cell complexes. Suppose there is a nonadjacency preserving map* $f\colon \mathcal{C}\to\mathcal{D}$*. If there is a symmetric* n*-cycle* z *of* $D(\mathcal{C})$ *such that* $I(z)=1$*, then there is symmetric* n*-cycle* w *of* $D(\mathcal{D})$ *such that* $I(w)=1$*.*

We use the next two lemmas in Theorem 18.

Lemma 16. *If a graph* G *has at least two vertices, then there is a symmetric* 0*-cycle* z *of* $D(G)$ *such that* $I(z)=1$*.*

Proof. Let v and w are distinct vertices of G. Then $z=v\times w+w\times v$ is a symmetric 0-cycle of $D(G)$ and $I(z)=1$. ∎

Lemma 17. *There are symmetric* 1*-cycles* w *and* z *of* $D(K_3)$ *and* $D(K_{1,3})$, *respectively, with* $I(w)=1$ *and* $I(z)=1$.

Proof. Label the vertices of K_3 as v_1,v_2,v_3 and the edges as e_1,e_2,e_3, such that for $i=1,2,3$, e_i does not have v_i as an end. Define $c=\sum_{i=1}^{3} v_i\times e_i$ and $w=c+T_1(c)$. It is easy to check that z is a symmetric 1-cycle and that $d_1(c)=f+T_0(f)$, where $f=v_1\times v_2+v_1\times v_3+v_2\times v_3$. Since the size of the support of f equals 3, we see that $I(w)=I(d_1(c))=1$.

Label the vertices of $K_{1,3}$ as v_0,v_1,v_2,v_3 and the edges as e_1,e_2,e_3, such that v_0 is the vertex of degree three and v_i is an end of e_i for $i=1,2,3$. Define $c=v_1\times e_2+v_1\times e_3+v_2\times e_3+v_2\times e_1+v_3\times e_1+v_3\times e_2$ and $z=c+T_1(c)$. It is to check that z is a symmetric 1-cycle and that $d_1(c)=f+T_0(f)$, where $f=v_1\times v_2+v_1\times v_3+v_2\times v_3$. Since the size of the support of f equals 3, we see that $I(z)=I(d_1(c))=1$. ∎

We can now complete the theorems presented in Section 3.

Theorem 18. *Let* $\mathcal{C}$ *be a cell complex whose underlying topological space is a simplicial complex. Then* $I(z)=0$ *for each symmetric* n*-cycle* z *of* $D(\mathcal{C})$ *if and only there is an even map* $\phi\colon \mathcal{C}\to\mathbb{R}^n$.

Proof. If there is an even map $\phi\colon \mathcal{C}\to\mathbb{R}^n$, then $I(z)=I(R(\phi),z)=0$.

Conversely, assume that $I(z)=0$ for each symmetric n-cycle z of $D(\mathcal{C})$. Let $\phi\colon \mathcal{C}\to\mathbb{R}^n$ be a proper continuous map. Then $J_\phi(\tilde{z})=0$ for each n-cycle of $\widetilde{D(\mathcal{C})}$. By Theorem 6 the theorem is valid for $n>1$. It remains to show for $n=0,1$ that $I(z)=0$ for each symmetric n-cycle z of $D(G)$ implies that there is an even map $\phi\colon G\to\mathbb{R}^n$.

Suppose $I(z)=0$ for each symmetric 0-cycle z of $D(G)$. Then G contains at most one vertex by Lemma 16. Hence there is an even map $\phi\colon G\to\mathbb{R}^0$.

Suppose $I(z)=0$ for each symmetric 1-cycle z of $D(G)$. Then G contains no $K_{1,3}$- or K_3-minor, as $D(K_{1,3})$ and $D(K_3)$ have a symmetric 1-cycle z with $I(z)=1$, by previous lemma. By Lemma 15 this implies that $D(G)$ has a symmetric 1-cycle w with $I(w)=1$. So G is a subgraph of a path. Hence there is an even map $\phi\colon G\to\mathbb{R}$. ∎

5. Closures of graphs

If $\phi\colon \mathcal{D}\to\mathbb{R}^n$ is a proper map and $f\colon \mathcal{C}\to\mathcal{D}$ is a nonadjacency preserving map, then $R(\phi\circ f)=R(\phi)\circ D(f)$. Hence, if $f\colon \mathcal{C}\to\mathcal{D}$ is a nonadjacency preserving map and $\phi\colon \mathcal{D}\to\mathbb{R}^n$ is an even map, then $R(\phi\circ f)_n(c)=$

$R(\phi)_n(D(f)_n(c))$ is trivial for each n-chain c of $D(\mathcal{C})$, so $\phi\circ f\colon \mathcal{C}\to\mathbb{R}^n$ is an even map. The question we can now pose is whether there exists a cell complex $\mathcal{D}$ with 1-skeleton equal to a graph G such that for each cell complex $\mathcal{C}$ with 1-skeleton equal to G, there exists a nonadjacency preserving map $f\colon \mathcal{C}\to\mathcal{D}$. Such a cell complex does indeed exist and we call it a closure of G. Then, if $\mathcal{D}$ has an even mapping in $\mathbb{R}^n$, each cell complex $\mathcal{C}$ with 1-skeleton equal to G has an even mapping in $\mathbb{R}^n$.

Let $\mathcal{C}$ be a cell complex. If σ_1 and σ_2 are cells of $\mathcal{C}$ and σ_1 is contained in the smallest subcomplex of $\mathcal{C}$ containing σ_2, we say that σ_1 is *incident* to σ_2. For $U\subseteq\mathcal{C}^0$, we denote by $\mathcal{C}[U]$ the subcomplex of $\mathcal{C}$ by deleting all cell incident to $\mathcal{C}^0\setminus U$.

For a graph $G=(V,E)$, a *closure* of G is a cell complex $\mathcal{C}$ such that

1. $\mathcal{C}^1$ is equal to G; and
2. for each integer $i\geq 0$ and each $U\subseteq V$ that induces a connected subgraph of G, $\pi_i(\mathcal{C}^{i+1}[U])$ is trivial.

This last condition can also be stated as: for each integer $i\geq 0$ and each $U\subseteq V$ that induces a connected subgraph of G, each continuous map $f\colon S^i\to\mathcal{C}^{i+1}[U]$ can be extended to a continuous map $F\colon B^{i+1}\to\mathcal{C}^{i+1}[U]$.

Theorem 19. *Each graph $G=(V,E)$ has a closure. Furthermore, we may assume that the underlying topological space of the closure is a simplicial complex.*

Proof. We recursively construct a closure $\mathcal{C}$. For $\mathcal{C}^1$ we take G.

Let $n\geq 1$ and suppose that we have constructed $\mathcal{C}^n$ such that for each $i=1,2,\dots,n-1$ and each $U\subseteq V$ that induces a connected subgraph of G, $\pi_i(\mathcal{C}^{i+1}[U])$ is trivial. Then we construct $\mathcal{C}^{n+1}$ as follows. For each $U\subseteq V$, $\pi_n(\mathcal{C}^n[U])$ is finitely generated, as $\mathcal{C}^n$ is a finite cell complex. Let $\alpha\colon S^n\to\mathcal{C}^n$ be a generator of $\pi_n(\mathcal{C}^n[U])$. By the cellular approximation theorem, we may assume that α is a cellular map. Attach an $(n+1)$-cell to $\mathcal{C}^n$ using α as attaching map. Repeating this for each subset $U\subseteq V$ and for each generator of $\pi_n(\mathcal{C}^n[U])$, we obtain a finite cell complex $\mathcal{C}^{n+1}$ with 1-skeleton equal to G and $\pi_n(\mathcal{C}^{n+1}[U])$ trivial for each $U\subseteq V$.

By taking the generators α simplicial, we may assume that $\mathcal{C}$ is a cell complex whose underlying topological space is a simplicial complex. ∎

Let $G=(V,E)$ and $H=(W,F)$ be graphs. We denote the set of all paths in H by $P(H)$. A function $\phi\colon V\cup E\to W\cup P(H)$ is an *immersion* if

1. $\phi(V)\subseteq W$ and $\phi(E)\subseteq P(H)$;
2. if e has ends v,w, then $\phi(e)$ is a path connecting $\phi(v)$ and $\phi(w)$; and
3. if e and f are nonadjacent edges, then $\phi(e)$ and $\phi(f)$ are disjoint paths.

Examples of immersions are the following:

- If G is a minor of H, then there is an immersion of G in H.
- If H is obtained from G by a ΔY-transformation, then there is an immersion from G in H.

Lemma 20. *Let $G=(V,E)$ and $H=(W,F)$ be graphs, and let $P(H)$ denote the set of all paths in H. Let $\mathcal{C}$ be a cell complex whose* 1*-skeleton is equal to G and let $\mathcal{D}$ be a closure of H. If there is an immersion $\phi\colon V\cup E\to W\cup P(H)$, then there is a nonadjacency preserving cellular map $g\colon \mathcal{C}\to\mathcal{D}$.*

Proof. We recursively define $g\colon \mathcal{C}\to\mathcal{D}$. First we define g on $\mathcal{C}^1$ by $g(v)=\phi(v)$ and $g(e)=\phi(e)$ for each vertex v and edge e of G.

Let $n\geq 1$ and assume that we have defined g on $\mathcal{C}^n$ such that for each pair of nonadjacent cells σ_1,σ_2 of $\mathcal{C}^n$, $g(\sigma_1)$ and $g(\sigma_2)$ are nonadjacent. We extend g to $\mathcal{C}^{n+1}$ such that whenever σ_1 and σ_2 are nonadjacent cells of $\mathcal{C}^{n+1}$, $g(\sigma_1)$ and $g(\sigma_2)$ are nonadjacent. To this end, let σ be an $(n+1)$-cell of $\mathcal{C}$ and let f_σ be its attaching map. Then f_σ is a mapping from $\partial B^{n+1}=S^n$ to $\mathcal{C}$. The composition $g\circ f_\sigma$ is a mapping from ∂B^{n+1} to $\mathcal{D}$. Let U be the set of vertices incident to $g\circ f_\sigma(\partial B^{n+1})$. Since $\mathcal{D}^{n+1}[U]$ is n-connected, we can extend $g\circ f_\sigma$ to B^{n+1} in $\mathcal{D}^{n+1}[U]$. Repeating this for each $(n+1)$-cell of $\mathcal{C}$, we obtain an extension of f to $\mathcal{C}^{n+1}$ such that $g(\sigma_1)$ and $g(\sigma_2)$ are nonadjacent for each pair of nonadjacent $(n+1)$-cells σ_1,σ_2. ∎

Corollary 21. *Let G be a graph and let $\mathcal{C}$ and $\mathcal{D}$ be closures of G. Then there is an adjacency preserving cellular map from $\mathcal{C}$ to $\mathcal{D}$.*

Corollary 22. *Let G be a graph containing a triangle and let H be obtained from G by a ΔY-transformation. Then there is a nonadjacency preserving cellular map from any closure of G to any closure of H.*

Corollary 23. *Let H be a graph and let G be a minor of H. Then there is a nonadjacency preserving cellular map from any closure of G to any closure of H.*

6. The graph parameter $\sigma(G)$

Let G be a graph and let $\mathcal{C}$ be a closure of G. From Lemma 20, it follows that any cell complex $\mathcal{D}$ whose 1-skeleton is equal to G has a nonadjacency preserving mapping into $\mathcal{C}$. If $\mathcal{C}$ has an even mapping into $\mathbb{R}^n$, or equivalent, if $I(z)=0$ for each symmetric n-cycle z of $D(\mathcal{C})$, then also $\mathcal{D}$ has an even mapping in $\mathbb{R}^n$. In particular, if $\mathcal{D}$ is a closure of G, then $\mathcal{C}$ has an even mapping into $\mathbb{R}^n$ if and only if $\mathcal{D}$ has an even mapping into $\mathbb{R}^n$. This leads us to define the graph parameter $\sigma(G)$.

For a graph G, we define $\sigma(G)$ as the smallest nonnegative integer n such that any closure $\mathcal{C}$ of G has an even mapping into $\mathbb{R}^n$. Equivalently, $\sigma(G)$ is the smallest integer $n \geq 0$ such that $I(z)=0$ for each symmetric n-cycle z of $D(\mathcal{C})$. Notice that $\mathcal{C}$ has an even mapping in $\mathbb{R}^{|G|}$, so $\sigma(G) \leq |G|$.

From Lemma 12 it follows that if $n \geq \sigma(G)$, then $I(z) = 0$ for each symmetric n-cycle z of $D(\mathcal{C})$. Hence, if a closure $\mathcal{C}$ of G has an even mapping in $\mathbb{R}^n$, it has also one in $\mathbb{R}^{n+1}$.

Theorem 24. *If there is an immersion of H in G, then $\sigma(H) \leq \sigma(G)$.*

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be closures of G and H, respectively. By Lemma 20, there is a nonadjacency preserving cellular map $g\colon \mathcal{D} \to \mathcal{C}$. Let $n=\sigma(G)$ and suppose that $n<\sigma(H)$. Then there is a symmetric n-cycle z of $D(\mathcal{D})$ such that $I(z)=1$. By Lemma 14, $I(D(g)_n(z))=I(z)=1$. The symmetric n-cycle $D(g)_n(z)$ shows that $n<\sigma(G)$. This contradiction shows that $\sigma(G) \leq \sigma(H)$. ∎

As corollaries we obtain:

Corollary 25. *If H is a minor of G, then $\sigma(H) \leq \sigma(G)$.*

Therefore, by the well-quasi-ordering theorem of Robertson and Seymour [13], the class of graphs G with $\sigma(G) \leq k$, with $k \geq 0$ an integer, can be described in terms of a finite number of forbidden minors. In Section 7, we characterize the classes of graphs G with $\sigma(G) \leq k$ for $k=0,1,\ldots,4$.

Corollary 26. *If H is obtained from G by a ΔY-transformation, then $\sigma(G) \leq \sigma(H)$.*

For a graph $G=(V,E)$, the *cone* of G, denoted $S(G)$, is the graph obtained from G by adding a new vertex adjacent to each vertex in G. We show now that $\sigma(S(G))=\sigma(G)+1$, except if G is the complement of K_2.

Let $\mathcal{I}$ be the cell complex with cells $\{0\},\{1\},(0,1)$. For a cell complex $\mathcal{C}$, we define the *cone* of $\mathcal{C}$ as the cell complex $(\mathcal{C} \times \mathcal{I})/(\mathcal{C} \times \{0\})$ (this is the cell complex obtained from $\mathcal{C} \times \mathcal{I}$ by identifying $\mathcal{C} \times \{0\}$ to one point). Notice that the 1-skeleton of the cone of a closure of a graph G is equal to $S(G)$, and that the cone of a closure of G is a closure of $S(G)$.

Lemma 27. *Let $\mathcal{C}$ be a closure of a graph G and let $\mathcal{D}$ be the cone of $\mathcal{C}$. If $n>0$, then for each symmetric n-cycle z of $D(\mathcal{C})$, there exists an $(n+1)$-chain f of $D(\mathcal{D})$ such that $d_{n+1}(f)=z$. Conversely, for each symmetric $(n+1)$-cycle w of $D(\mathcal{D})$, there exists an $(n+1)$-chain f such that $w=f+T_{n+1}(f)$ and $d_{n+1}(f)$ is a symmetric n-cycle of $D(\mathcal{C})$.*

Proof. Let p be the unique 0-cell of $\mathcal{D}$ which is not in $\mathcal{C}$. For each cell σ of $\mathcal{C}$, define $C(\sigma)$ to be the unique cell in the cone of σ of dimension $\dim\sigma+1$, and extend C linearly to chains of $\mathcal{C}$. Note that $d_{k+1}(C(\sigma))=\sigma+C(d_k(\sigma))$ if $k=\dim\sigma>0$ and that $d_1(C(\sigma))=\sigma+p$ if $\dim\sigma=0$.

For each nonnegative integer n, define $\rho_n\colon \overline{C}_n(D(\mathcal{C}))\to\overline{C}_{n+1}(D(\mathcal{D}))$ on each n-cell $\sigma\times\tau$ by

$$\rho_n(\sigma\times\tau)=C(\sigma)\times\tau,$$

and extend ρ_n linearly to $\overline{C}_n(D(\mathcal{C}))$. For any n-cell $\sigma\times\tau$ of $D(\mathcal{C})$ with $k=\dim\sigma>0$, we have

$$\begin{aligned} d_{n+1}(\rho_n(\sigma\times\tau)) &= \sigma\times\tau+C(d_k(\sigma))\times\tau+C(\sigma)\times d_{n-k}(\tau)\\ &= \sigma\times\tau+\rho_{n-1}(d_n(\sigma\times\tau)), \end{aligned}$$

and for any n-cell $\sigma\times\tau$ of $D(\mathcal{C})$ with $\dim\sigma=0$, we have

$$\begin{aligned} d_{n+1}(\rho_n(\sigma\times\tau)) &= p\times\tau+\sigma\times\tau+C(\sigma)\times d(\tau)\\ &= p\times\tau+\sigma\times\tau+\rho_{n-1}(d_n(\sigma\times\tau)). \end{aligned}$$

Let z be a symmetric n-cycle of $D(\mathcal{C})$. We can write $z=g+h$, where g is an n-chain containing only cells $\sigma\times\tau$ with $\dim\sigma>0$ and h is an n-chain containing only cells $\sigma\times\tau$ with $\dim\sigma=0$. Then $\rho_n(z)=\rho_n(g)+\rho_n(h)$ and

$$\begin{aligned} d_{n+1}(\rho_n(z)) &= d_{n+1}(\rho_n(g))+d_{n+1}(\rho_n(h))\\ &= g+\rho_{n-1}(d_n(g))+p\times c+h+\rho_{n-1}(d_n(h))\\ &= p\times c+z, \end{aligned}$$

where c is an n-chain of $\mathcal{C}$. Since $d_{n+1}(\rho_n(z))$ and z are n-cycles, $p\times c$ is an n-cycle of $D(\mathcal{C})$ (which is not necessarily symmetric), so c is an n-cycle of $\mathcal{C}$. Since $\mathcal{C}$ is a closure, there exists an $(n+1)$-chain b of $\mathcal{C}$ such that $d_{n+1}(b)=c$. Define

$$f=\rho_n(z)+p\times b.$$

Then

$$\begin{aligned} d_{n+1}(f) &= d_{n+1}(\rho_n(z))+d_{n+1}(p\times b)\\ &= p\times c+z+p\times c=z. \end{aligned}$$

Conversely, let w be a symmetric $(n+1)$-cycle of $D(\mathcal{D})$. We can write $w=g+T_{n+1}(g)+h+T_{n+1}(h)$, where g contains cells $\sigma\times\tau$ with p incident with σ and where h contains cells $\sigma\times\tau$ with p incident with neither σ nor τ. Let k be an $(n+1)$-chain of $\mathcal{C}$ such that $g+p\times k$ contains no cells of the form $p \times \tau$ with τ an $(n+1)$-cell of $\mathcal{C}$. We can write $g+p\times k=\rho_n(j)$ for some n-chain j of $D(\mathcal{C})$. Then

$$d_{n+1}(g+p\times k)=d_{n+1}(\rho_n(j))=j+\rho_{n-1}(d_n(j))+p\times l$$

for an n-chain l of $\mathcal{C}$. Hence

$$d_{n+1}(g)=j+\rho_{n-1}(d_n(j))+p\times l+p\times d_{n+1}(k).$$

Since w is an $(n+1)$-cycle of $D(\mathcal{D})$, $p\times l+p\times d_{n+1}(k)=0$ and $\rho_{n-1}(d_n(j))=0$, and so $d_{n+1}(g)=j$. Hence $d_{n+1}(g+h)$ is an n-cycle of $D(\mathcal{C})$, and since $g+h+T_n(g+h)$ is a symmetric $(n+1)$-cycle, $d_{n+1}(g+h)$ is a symmetric n-cycle of $D(\mathcal{C})$. ∎

Theorem 28. *Let G be a graph. Then $\sigma(S(G))=\sigma(G)+1$ unless G contains exactly two vertices and no edges.*

Proof. Let $\mathcal{C}$ be closures of G, and let $\mathcal{D}$ be the cone of $\mathcal{C}$. We first show that $\sigma(S(G))\geq\sigma(G)+1$. Let $n=\sigma(G)$. For $n=0$, the inequality holds, as $S(G)$ contains at least two vertices, and hence $\sigma(S(G))>0$. If $n=1$ and G contains at least three vertices, then $S(G)$ has a subgraph isomorphic to $K_{1,3}$. By Corollary 25, $\sigma(S(G))\geq\sigma(K_{1,3})=2$. If $n=1$ and G is isomorphic to K_2, then $S(G)$ is isomorphic to K_3. Hence $\sigma(S(G))=2$. For $n\geq 2$, we use that there exists a symmetric $(n-1)$-cycle z of $D(\mathcal{C})$ such that $I(z)=1$. By Lemma 27, there exists an n-chain c of $D(\mathcal{D})$ such that $d_n(c)=z$. Then

$$I(c+T_n(c))=I(d_n(c))=I(z)=1,$$

by Lemma 12. So $\sigma(S(G))\geq n+1=\sigma(G)+1$.

We next show that $\sigma(S(G))\leq\sigma(G)+1$. Let $n=\sigma(G)$ and let z be an arbitrary symmetric $(n+1)$-cycle of $D(\mathcal{D})$. By Lemma 27, there exists an $(n+1)$-chain c of $D(\mathcal{D})$ such that $c+T_{n+1}(c)=z$ and $d_{n+1}(c)$ is a symmetric n-cycle of $D(\mathcal{C})$. Since $n=\sigma(G)$, $I(d_{n+1}(c))=0$. So

$$I(z)=I(c+T_{n+1}(c))=I(d_{n+1}(c))=0,$$

by Lemma 12. Hence $\sigma(S(G))\leq\sigma(G)+1$. ∎

7. Characterizations

From Corollary 25, it follows that the class of graphs G with $\sigma(G)\leq k$ is closed under taking minors. We now characterize the classes of graphs G with $\sigma(G)\leq k$, for $k=0,1,\ldots,4$. For example, the class of graphs G with $\sigma(G)\leq 3$ is the class of planar graphs. One step in proving this, is to show that planar graphs G have $\sigma(G) \leq 3$. A planar graph G clearly has an even mapping in $\mathbb{R}^2$. Hence $I(z)=0$ for each symmetric 2-cycle z of $D(G)$. Let $\mathcal{C}$ be a closure of G. We now prove that for each symmetric n-cycle z of $D(\mathcal{C}^{\lfloor (n+1)/2 \rfloor})$, there exists a symmetric $(n+1)$-cycle $w=c+T_{n+1}(c)$ of $D(\mathcal{C})$ such that $d_{n+1}(c)=z$. (For a rational number α, $\lfloor \alpha \rfloor$ is the largest integer no larger than α.) In particular, for each symmetric 2-cycle of $D(G)$, there exists a symmetric 3-cycle $w=c+T_3(c)$ of $D(\mathcal{C})$ such that $d_3(c)=z$. So $\sigma(G) \leq 3$ by Lemma 12.

Lemma 29. *Let $n>1$ be an integer and let $k=\lfloor (n+1)/2 \rfloor$. Let $\mathcal{C}$ be a closure of a graph G. Then, for each symmetric n-cycle z of $D(\mathcal{C}^k)$, there exists an $(n+1)$-chain c of $D(\mathcal{C})$ such that $d_{n+1}(c)=z$. Conversely, for each symmetric $(n+1)$-cycle w of $D(\mathcal{C})$, there exists an $(n+1)$-chain c of $D(\mathcal{C})$ such that $w=c+T_{n+1}(c)$ and $d_{n+1}(c)$ is a symmetric n-cycle of $D(\mathcal{C}^k)$.*

Proof. We first show that for each symmetric n-cycle z of $D(\mathcal{C}^k)$, there exists an $(n+1)$-chain c of $D(\mathcal{C})$ such that $d_{n+1}(c)=z$. To this end, let t be the largest integer for which there exists an $(n+1)$-chain c of $D(\mathcal{C})$ such that $w=z-d_{n+1}(c)$ is an n-cycle of $D(\mathcal{C})$ containing no cells of the form $\sigma\times\tau$ with $\dim\sigma<t$. If $t>n$, then $w=0$, and so $d_{n+1}(c)=z$. Suppose now for a contradiction that $t\leq n$.

For each $(n-t)$-cell τ of $\mathcal{C}$, let $Z(w,\tau)$ be the t-cycle of $\mathcal{C}$ consisting of all t-cells σ of $\mathcal{C}$ such that $\sigma\times\tau$ occurs in w. Let U be the set of vertices incident with any t-cell in $Z(w,\tau)$. Since $\mathcal{C}^{(t+1)}[U]$ is t-connected, there is a $(t+1)$-chain $C(w,\tau)$ of $\mathcal{C}^{(t+1)}[U]$ such that $d_{t+1}(C(w,\tau))=Z(w,\tau)$. Let $c'=\sum_\tau C(w,\tau)\times\tau$, where the sum is over all $(n-t)$-cells τ of $\mathcal{C}$. Since each vertex in U is not incident with τ, c' is a $(k+1)$-chain of $D(\mathcal{C})$. Then $z-d_{n+1}(c+c')$ is an n-cycle containing no cells of the form $\sigma\times\tau$ with $\dim\sigma<t+1$, contradicting the assumption on t.

We next show that for each symmetric $(n+1)$-cycle w of $D(\mathcal{C})$, there exists an $(n+1)$-chain c of $D(\mathcal{C})$ such that $w=c+T_{n+1}(c)$ and $d_{n+1}(c)$ is a symmetric n-cycle of $D(\mathcal{C}^k)$. To this end, let w be an $(n+1)$-cycle of $D(\mathcal{C})$. There is an $(n+1)$-chain c containing only cells $\sigma\times\tau$ with $\dim\sigma\leq\dim\tau$ such that $w=c+T_{n+1}(c)$. Then $d_{n+1}(c)$ is a symmetric n-cycle that contains only cells $\sigma'\times\tau'$ with $\dim\sigma'+\dim\tau'=n$ and $\dim\sigma'\leq\dim\tau'+1$. From $\dim\sigma'\leq\dim\tau'+1$ it follows that $2\dim\sigma'\leq n+1$, and so $\dim\sigma'\leq k$. Since $d_{n+1}(c)$ is a symmetric n-cycle, also $\dim\tau'\leq\dim\sigma'+1$, and so $\dim\tau'\leq k$. Hence $d_{n+1}(c)$ is a symmetric n-cycle of $D(\mathcal{C}^k)$. ∎

Theorem 30. *Let $n>1$ be an integer and let $k=\lfloor (n+1)/2 \rfloor$. Let $\mathcal{C}$ be a closure of a graph G. Then $I(z)=0$ for each symmetric n-cycle z of $D(\mathcal{C}^k)$ if and only if $I(w)=0$ for each symmetric $(n+1)$-cycle w of $D(\mathcal{C})$.*

Proof. Let w be a symmetric $(n+1)$-cycle of $D(\mathcal{C})$ such that $I(w)=1$. By Lemma 29, there exists an $(n+1)$-chain c of $D(\mathcal{C})$ such that $w=c+T_{n+1}(c)$ and $z=d_{n+1}(c)$ is a symmetric n-cycle of $D(\mathcal{C}^k)$. Then $I(z)=I(d_{n+1}(c))=I(w)=1$.

Conversely, let z be a symmetric n-cycle of $D(\mathcal{C}^k)$ such that $I(z)=1$. By Lemma 29, there exists an $(n+1)$-chain c of $D(\mathcal{C})$ such that $d_{n+1}(c)=z$. Let $w=c+T_{n+1}(c)$. Then $I(w)=I(d_{n+1}(c))=I(z)=1$. ∎

As $K_{3,3}$ has a mapping in $\mathbb{R}^2$ that has exactly one pair of nonadjacent edges with an odd intersection, we know that there is a symmetric 2-cycle z of $D(G)$ with $I(z)=1$, that is, G has no even mapping in $\mathbb{R}^2$. Another way of verifying that $K_{3,3}$ has no even mapping in $\mathbb{R}^2$ is by considering the symmetric 2-cycle $z=\sum e\times f$ of $D(G)$ with the sum ranging over all ordered pairs of nonadjacent edges of $K_{3,3}$. Applying Lemma 12 twice shows that $I(z)=1$. By Theorem 30, $\sigma(K_{3,3})>3$. To see that $\sigma(K_{3,3})\leq 4$, let $\mathcal{C}$ be a closure of $K_{3,3}$. Since the only symmetric 3-cycle in $D(\mathcal{C}^2)$ is the zero cycle, $I(z)=0$ for every symmetric 3-cycle z of $D(\mathcal{C}^2)$. By Theorem 30, $I(w)=0$ for every symmetric 4-cycle w of $D(\mathcal{C})$. Hence $\sigma(K_{3,3})=4$, and in the same way it can be shown that $\sigma(K_5)=4$.

Lemma 31. *$\sigma(K_3)>1$, $\sigma(K_{1,3})>1$, $\sigma(K_4)>2$, and $\sigma(K_{2,3})>2$.*

Proof. From Lemma 17, it follows that $\sigma(K_3)>1$ and that $\sigma(K_{1,3})>1$.

If $\sigma(K_{2,3})\leq 2$, then $\sigma(K_{3,3})\leq 3$ by Theorem 28 and Corollary 25. Hence $\sigma(K_{2,3})>2$. Similarly, $\sigma(K_4)>2$. ∎

Theorem 32. *A graph G contains exactly one vertex if and only if $\sigma(G)=0$.*

Proof. This follows immediately from Lemma 16. ∎

Theorem 33. *A graph G is a disjoint union of paths if and only if $\sigma(G)\leq 1$.*

Proof. If G is a disjoint union of paths, it has an embedding in the line. As each pair of nonadjacent edge and vertex has no intersection, $\sigma(G)\leq 1$.

Conversely, if G is not a disjoint union of paths, then G has a K_3- or $K_{1,3}$-minor. Since, $\sigma(K_3)>1$ and $\sigma(K_{1,3})>1$, $\sigma(G)>1$. ∎

Theorem 34. *A graph G is outerplanar if and only if $\sigma(G)\leq 2$.*

Proof. If G is outerplanar, it has an embedding in the plane such that each circuit bounds a disc with no vertices in its interior. Hence each pair of nonadjacent edges, and each pair of nonadjacent 2-cell and vertex has no intersection. Hence $\sigma(G)\leq 2$.

Conversely, if G is not outerplanar, then G has a K_4- or a $K_{2,3}$-minor. Since $\sigma(K_4)>2$ and $\sigma(K_{2,3})>2$, $\sigma(G)>2$. ∎

Theorem 35. *A graph G is planar if and only if $\sigma(G) \leq 3$.*

Proof. Let $\mathcal{C}$ be a closure of G. If G is planar, then the embedding of G in $\mathbb{R}^2$ shows that G has an even mapping in $\mathbb{R}^2$. By Theorem 30, $\mathcal{C}$ has an even mapping in $\mathbb{R}^3$. Hence $\sigma(G) \leq 3$.

Conversely, if G is not planar, then, by Kuratowski's theorem, G has a $K_{3,3}$- or K_5-minor. Since $\sigma(K_{3,3}) = 4$ and $\sigma(K_5) = 4$, Corollary 25 shows that $\sigma(G) \geq 4$. ∎

The *Petersen family* is the collection of graph obtained from K_6 and $K_{1,3,3}$ by applying ΔY-transformations. By Robertson, Seymour, and Thomas [12], the Petersen family is the complete collection of forbidden minors of graphs that have a flat embedding.

Theorem 36. *A graph G has a flat embedding if and only if $\sigma(G) \leq 4$.*

Proof. Let $\mathcal{C}$ be a closure of G. If G has a flat embedding in $\mathbb{R}^3$, then for each circuit C of G, there is an open disc embedded in $\mathbb{R}^3$ with boundary C, disjoint from G. For each 2-cell σ of $\mathcal{C}$, we map σ onto the open disc bounded by the circuit on the boundary of σ. This yields a mapping of $\mathcal{C}^2$ into $\mathbb{R}^3$ such that for each nonadjacent pair of 2-cell and edge, the intersection number is zero mod 2. So there is an even mapping of $\mathcal{C}^2$ into $\mathbb{R}^3$. By Theorem 30, $\sigma(G) \leq 4$.

Conversely, if G has no flat embedding, then G contains a minor isomorphic to one of the graphs in the Petersen family. Since $\sigma(K_{3,3}) = 4$, we know by Theorem 28 that $\sigma(K_{1,3,3}) = 5$. The graphs in the Petersen family can be obtained from K_6 and $K_{1,3,3}$ by applying ΔY-transformation. Since $\sigma(K_6) = 5$ and $\sigma(K_{1,3,3}) = 5$, we obtain by Corollary 26 that each graph H in the Petersen family has $\sigma(H) > 4$. By Corollary 25, $\sigma(G) > 4$. ∎

Two affine subspaces H and H' of $\mathbb{R}^d$ are *parallel* if their projective hulls have a nonempty intersection that is contained in the hyperplane at infinity. We call two faces F and F' of a full-dimensional polytope P *parallel* if their affine spans are parallel. Two faces F and F' of P are *antipodal* if there exists a nonzero vector $c \in \mathbb{R}^d$ such that the linear function $c^T x$ is maximized by every point of F and minimized by every point of F'.

If P is a full-dimensional polytope in $\mathbb{R}^n$, then $-P$ denotes the polytope $\{-x \mid x \in P\}$. The *Minkowski sum*, $P+Q$, of two polytopes P and Q in $\mathbb{R}^n$ is the polytope $\{x+y \mid x \in P, y \in Q\}$. In particular, $P-P$ is the polytope $\{x-y \mid x, y \in P\}$. If P has no parallel faces, then each face F of $\partial(P-P)$ can uniquely be written as $F_1 - F_2$, where F_1 and F_2 are antipodal faces of ∂P. Moreover, each point p in $\partial(P-P)$ can uniquely be written as $p = p_1 - p_2$, where $p_1, p_2 \in \partial P$.

Lemma 37. *Let G be a graph and let $P \subseteq \mathbb{R}^{n+1}$ be a full-dimensional polytope with no parallel faces. If there is a cellular map $\phi\colon P^1 \to G$ such that for every pair of antipodal faces F_1, F_2 of P, the smallest subgraphs of G containing $\phi(F_1^1)$ and $\phi(F_2^1)$, respectively, have no common vertices, then $\sigma(G) > n$.*

Proof. Let $\mathcal{C}$ be a closure of G. We recursively construct a cellular mapping Φ of P into $\mathcal{C}$. Define the restriction of Φ to P^1 to be ϕ and assume that we have defined Φ on the t-skeleton of P. Consider a $(t+1)$-face F of P. Let U be the set of vertices in $\Phi(\partial F)$ (this is equal to the set of vertices of $\Phi(F^1)$). Since $\mathcal{C}^{t+1}[U]$ is t-connected, Φ can be extended to a continuous map from F into $\mathcal{C}^{t+1}[U]$. Doing this for each $(t+1)$-face F of P, we obtain a map Φ from the $(t+1)$-skeleton of P into $\mathcal{C}^{t+1}$. By construction, $\Phi(F_1)$ and $\Phi(F_2)$ are nonadjacent if F_1 and F_2 are antipodal faces of P.

Let M be the Minkowski sum of P and $-P$. Define $T(x) = -x$ for each $x \in \partial M$. Then $(\partial M, T)$ is an antipodal cell complex. Define $h\colon \partial M \to S^n$ by $h(p) = p/\|p\|$. By Lemma 8, there exists an equivariant cellular map $g\colon (\mathcal{S}^n, R) \to (\partial M, T)$ homotopic to h^{-1}. Since P has no pair of parallel faces, each point p in ∂M can uniquely be written as $p = p_1 - p_2$, where $p_1, p_2 \in \partial P$. Define $f\colon \partial M \to D(\mathcal{C})$ by $f(p) = (\Phi(p_1), \Phi(p_2))$, where p_1 and p_2 are the unique points of ∂P such that $p = p_1 - p_2$. Then f is an equivariant cellular map.

Let $w = f_n(g_n(s_n))$. By Theorem 13 and Lemma 14, $I(w) = I(g_n(s_n)) = I(s_n) = 1$. Hence $\sigma(G) > n$. ∎

8. The Colin de Verdière number

Let $G = (V, E)$ be a graph with $V = \{1, 2, \ldots, n\}$ and let O_G be the collection of all symmetric $n \times n$ matrices $M = (m_{i,j})$ with

1. $m_{i,j} < 0$ if i and j are connected by an edge, and
2. $m_{i,j} = 0$ if $i \neq j$, and i and j are not connected by an edge.

(So the entries on the diagonal may be any real number.) A matrix $M \in O_G$ fulfills the *Strong Arnol'd Property* if the all-zero matrix is the only symmetric matrix $X = (x_{i,j})$ that satisfies $x_{i,j} = 0$ if $i = j$ or if i and j are adjacent, and $MX = 0$. The parameter $\mu(G)$ is defined as the largest corank of any matrix $M \in O_G$ with exactly one negative eigenvalue, and fulfilling the Strong Arnol'd Property.

The parameter $\mu(G)$ has the property that $\mu(G') \leq \mu(G)$ if G' is a minor of G; see [2,3]. Hence by the well-quasi-ordering theorem of Robertson and

Seymour, the class of all graph G with $\mu(G) \le k$ can be described in terms of a finite collection of forbidden minors. The following characterizations are known:

1. $\mu(G) = 0$ if and only if G consists of only one vertex.
2. $\mu(G) \le 1$ if and only if G is a subgraph of a path.
3. $\mu(G) \le 2$ if and only if G is outerplanar.
4. $\mu(G) \le 3$ if and only if G is planar. (See Colin de Verdière [2,3] and see van der Holst [4] for a short proof.)
5. $\mu(G) \le 4$ if and only if G has a flat embedding. (See Lovász and Schrijver [8].)

Notice that for $k \le 4$, $\mu(G) \le k$ if and only if $\sigma(G) \le k$. For more information and theorems on the Colin de Verdière parameter, we refer to [7].

For any vector $x \in \mathbb{R}^n$, define $\text{supp}(x) = \{i \mid x_i \neq 0\}$, $\text{supp}_+(x) = \{i \mid x_i > 0\}$, $\text{supp}_-(x) = \{i \mid x_i < 0\}$, and $\text{supp}_0(x) = \{i \mid x_i = 0\}$. We call these sets, respectively, the support, the positive, negative, and null support of vector x. We call the triple of subsets $(\text{supp}_+(x), \text{supp}_-(x), \text{supp}_0(x))$ the *sign vector* of x.

Closely related to $\mu(G)$ is the parameter $\lambda(G)$. Call a linear subspace $L \subseteq \mathbb{R}^n$ a *valid representation* of G if for each nonzero $x \in L$, $G[\text{supp}_+(x)]$ is nonempty and connected. The graph parameter $\lambda(G)$ is defined as

$$\lambda(G) := \max\{\dim(L) \mid L \text{ is a valid representation of } G\}.$$

There are several theorems known about $\lambda(G)$ (see [6] and [10]), of which we mention only:

Theorem 38 **([10]).** *For all connected graphs G, $\mu(G) \le \lambda(G) + 2$.*

We will now show how to construct a polytope $P(L)$ from a valid representation L of G.

If s is a sign vector of x, then $(s)_+$, $(s)_-$ are defined to be $\text{supp}_+(x)$, $\text{supp}_-(x)$, respectively. For a linear subspace $L \subseteq \mathbb{R}^n$, we denote by $\mathcal{S}_L$ the set of sign vectors of all vectors in L. We put a partial ordering on $\mathcal{S}_L$ by defining $s \le t$ if $(s)_+ \subseteq (t)_+$ and $(s)_- \subseteq (t)_-$.

Let $x_1, x_2, \ldots, x_d$ be a basis of L and let $z_1, z_2, \ldots, z_n \in \mathbb{R}^d$ be defined by

$$\begin{pmatrix} x_1^T \\ \vdots \\ x_d^T \end{pmatrix} = \begin{pmatrix} z_1 & \ldots & z_n \end{pmatrix}.$$

Every vector $x \in L$ is of the form $x = (c^T z_1, c^T z_2, \ldots, c^T z_n)^T$ for some $c \in \mathbb{R}^d$. Let $\mathcal{A}$ be the hyperplane arrangement consisting of all hyperplanes orthogonal to some z_i, $i = 1, 2, \ldots, n$. This hyperplane arrangement partitions $\mathbb{R}^d$

into a set of open cones. If C is one of these open cones, then all vectors $(c^T z_1, c^T z_2, \ldots, c^T z_n)$ with $c \in C$ have the same sign vector. If C_1 and C_2 are distinct open cones and $c_1 \in C_1$ and $c_2 \in C_2$, then the sign vectors of $(c_1^T z_1, c_1^T z_2, \ldots, c_1^T z_n)$ and $(c_2^T z_1, c_2^T z_2, \ldots, c_2^T z_n)$ are different. Hence there is a 1-1 correspondence between this set of open cones and $\mathcal{S}_L$.

Let

$$Z = \left\{ \sum_{i=1}^{n} \lambda_i z_i \mid -1 \leq \lambda_i \leq +1 \right\}.$$

This is a zonotope and there is a 1-1 correspondence between the nonempty open faces of Z and the open cones partitioned by $\mathcal{A}$. Hence there is a 1-1 correspondence between the nonempty open faces of Z and the sign vectors of the linear space L. If s and t are sign vectors of L with $s<t$, and F and H are their corresponding faces of Z, then H is a face on the boundary of F. In particular, the minimal nonzero sign vectors s of $\mathcal{S}_L$ correspond to the facets of Z. (The zero sign vector corresponds to Z itself.)

Let $P(L)$ be the polar polytope of Z. The nonempty open faces of $P(L)$ correspond to the sign vectors of L. If F and H are nonempty open faces of $P(L)$ such that F belongs to the boundary of H, then their corresponding sign vectors s,t in $\mathcal{S}_L$ satisfy $s < t$. If F is a face of $P(L)$, we denote its corresponding sign vector in $\mathcal{S}_L$ by s_F. Since $P(L)$ is centrally symmetric, $-F$ is also a face if F is a face of $P(L)$. Moreover, $s_{-F} = -s_F$. If F and F' are antipodal faces of $P(L)$, then F and $-F'$ belong to boundary of a facet D of $P(L)$, and hence $s_F < s_D$ and $s_{-F'} < s_D$; that is, $(s_F)_+ \subseteq (s_D)_+$ and $(s_{F'})_+ \subseteq (s_D)_-$.

Lemma 39. *Let G be a graph and let L be a valid representation of G of dimension d. Then there is a cellular map ϕ of $P(L)^1$ into G such that for every pair of antipodal faces F_1, F_2 of $P(L)$, the smallest subgraph of G containing $\phi(F_1^1)$ and $\phi(F_2^1)$, respectively, have no common vertices.*

Proof. Each vertex p of $P(L)$ corresponds to a sign vector s_p of minimal support. Choose a vertex $v \in (s_p)_+$ and define $\phi(p) = v$. Each edge $e = pq$ of $P(L)$ corresponds to a sign vector s_e. Then $(s_p)_+ \subseteq (s_e)_+$ and $(s_q)_+ \subseteq (s_e)_+$. Since $G[(s_e)_+]$ is connected, there is a path $P(L)$ in $G[(s_e)_+]$ connecting vertex $\phi(p)$ to $\phi(q)$. Define ϕ restricted to e as a continuous mapping from e into G. Thus we have obtained a cellular mapping ϕ from $P(L)^1$ into G.

Let F_1, F_2 be a pair of antipodal faces of $P(L)$, and let s_{F_1} and s_{F_2} be their corresponding sign vectors. As F_1 and F_2 are antipodal faces, there exists a facet D of $P(L)$ such that F_1 and $-F_2$ belong to the boundary of D. Let s_D be a sign vector corresponding to D. Then $(s_{F_1})_+ \subseteq (s_D)_+$

and $(s_{-F_2})_+ \subseteq (s_D)_+$. This implies $(s_{F_1})_+ \subseteq (s_D)_+$ and $(s_{F_2})_+ \subseteq (s_D)_-$, and hence $\phi(F_1^1)$ and $\phi(F_2^1)$ are nonadjacent. ∎

Theorem 40. *For any graph G, $\lambda(G) \le \sigma(G)$.*

Proof. Let $d = \sigma(G)$ and let n be the number of vertices of G. Suppose for a contradiction that $\lambda(G) > \sigma(G)$. Then there exists a $(d+1)$-dimensional linear subspace $L \subseteq \mathbb{R}^n$ such that for every nonzero $x \in L$, the subgraph induced by $\text{supp}_+(x)$ is nonempty and connected. By Lemma 39, there is a cellular map $\phi\colon P(L)^1 \to G$ such that $\phi(F_1^1)$ and $\phi(F_2^1)$ are nonadjacent for each pair of antipodal faces F_1, F_2 of $P(L)$. By perturbing the polytope $P(L)$ in projective $(d+1)$-space (see [8] for a description of this perturbing process), we can find a polytope P in $\mathbb{R}^{d+1}$ with the same combinatorial structure as $P(L)$, but without parallel faces. By Lemma 37, $\sigma(G) > d$. This contradiction shows that $\lambda(G) \le \sigma(G)$. ∎

Using Theorem 38, we get:

Corollary 41. *For any graph G, $\mu(G) \le \sigma(G) + 2$.*

We pose the conjectures:

Conjecture 42. If G is a graph with $\sigma(G) \le d$, then $\mu(G) \le d$.

The converse statement of Conjecture 42 is false. In [10] a graph G is given that has $\mu(G) \le 18$, whereas $\lambda(G) \ge 20$, and so there is no even mapping of a 9-closure of G into 18-space. Nevertheless, we pose the following conjecture.

Conjecture 43. A graph G has $\mu(G) \le 5$ if and only if $\sigma(G) \le 5$.

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